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On the description of electrons on a noncommutative plane threaded by a transversal magnetic field

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Abstract

In this paper we deal with the quantum-mechanical description of electrons on a noncommutative plane under the influence of a transversal and homogeneous magnetic field. For this purpose the noncommutativity has been implemented by resorting to the quantum Euclidean space relying on the quantum group $SO_q(N)$ and to the so-called noncommutative θ-formulation, which is familiar in string theory. Matching conditions have been established and discussed.

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1. Introduction

A quantum-mechanical description of electrons on a noncommutative plane threaded by a transversal and homogeneous magnetic field has been discussed recently [1]. Here the noncommutative coordinates are introduced as

$$[x, y] = \mathrm{i}\theta \tag{1}$$

which proceeds in accordance with string-theory arguments [2]. In general, we have to assume that the noncommutativity parameter θ exhibits very small values such as, for example, $|\,\theta\,| \lesssim (10^4\ \mathrm{GeV})^{-2}$ [3]. A similar problem has been considered on the noncommutative torus [4]. Alternative descriptions [5] and relationships between the quantized Hall conductivity and the θ-parameter [6] are also worth mentioning. Such issues look promising, even if they are not an absolute novelty. Indeed, the quantization of the Hall conductivity has also been discussed before [7] by using the noncommutative geometry introduced by Connes [8]. Moreover, physical systems such as the hydrogen atom [9, 10] and the harmonic oscillator [11, 12] have been discussed in some detail on the noncommutative quantum Euclidean space. In this latter case, both commutation relations and the metric tensor are established by virtue of the $\boldsymbol{R}$-matrix description of the quantum group $SO_q(N)$, where N denotes the number of space dimensions [13]. A radial reduction of the covariant $SO_q(N)$-derivative has been done, which results in the onset of the Jackson derivative [14, 15]. Accordingly, well-defined q-deformed versions of the radial Schrödinger equations have been written down [16, 17]. Now the related deformation parameter is denoted by q, which has to be viewed (excepting generic cases in which q is a pure phase) as being very near to unity, i.e.

$$q = \exp \gamma \cong 1 + \gamma. \tag{2}$$

Now the question arises how to compare the θ- and q-noncommutative descriptions mentioned above. For this purpose we perform the description of electrons on the quantum Euclidean plane, which proceeds under the influence of a transversal magnetic field $\vec{B} = (0, 0, B)$. This opens the way to establishing quite a meaningful relationship between θ- and q-parameters, now in terms of the matching condition for the energy.

2. Noncommutative preliminaries and q-deformations

The commutation relations characterizing the three-dimensional quantum Euclidean space are given by [13]

$$x_1 x_2 = q x_2 x_1 \tag{3}$$

$$x_2 x_3 = q x_3 x_2 \tag{4}$$

and

$$x_1 x_3 - x_3 x_1 = \frac{1-q}{\sqrt{q}} x_2 x_2. \tag{5}$$

Inserting $x_1 = (x + \mathrm{i}y)/\sqrt{2}$ and $x_3 = (x - \mathrm{i}y)/\sqrt{2}$ into (5) gives

$$[x, y] = -2\mathrm{i} \sinh \frac{\gamma}{2} x_2 x_2 \tag{6}$$

in which x_2 corresponds to the z-coordinate. Comparing (1) and (6) yields the intermediary result

$$\theta = -2 \sinh \frac{\gamma}{2} x_2 x_2 \tag{7}$$

which also shows that $x_2 x_2$ remains to be fixed. Keeping in mind this conjecture, we then deal properly with electrons on a selected noncommutative quantum plane which is perpendicular to the direction of the magnetic field, i.e. with a problem in 2 + 1 space dimensions. Of course, equations (3)–(5) reproduce the usual commutative coordinates as soon as $q \rightarrow 1$. Such behaviours can be traced back to the noncommutative space-time proposed long ago by Snyder [18]. In this latter case, the usual coordinates are reproduced if the 'elementary length' tends to zero, which agrees with equation (1). The θ-parameter can be viewed, of course, as the square of related 'elementary length' l_0, i.e. $\theta = l_0^2$ or $\theta = -l_0^2$.

The radial coordinate is introduced by

$$r^2 = C_{ij} x^i x^j = x^i x_i \tag{8}$$

where C_{ij} is the metric tensor, so that $[r^2, x_i] = 0$ [13]. This enables us to perform the radial reduction of the covariant derivative ∂_i as

$$\partial_i = \frac{x_i}{r} \partial_r^{(q)} \tag{9}$$

in which case

$$\partial_r^{(q)} f(r) = \frac{\mu}{q+1} \frac{d_q f(r)}{d_q r} = \frac{\mu}{q+1} \frac{f(qr) - f(r)}{r(q-1)} \tag{10}$$

where $\mu = 1 + q^{2-N}$. We are then ready to define the $SO_q(N)$-deformed counterpart of the reduced radial Schrödinger equation as (see section 3 in [17])

$$\left[-\frac{q}{(q+1)^2}(q^L + q^{-L})^2 \partial_r^{(q)^2} - \frac{\mu^2 q^2}{(q+1)^4}[[-2l - N + 1]]_q\,[[2l + N - 3]]_q \frac{1}{r^2} + V(r)\right]\varphi_q(r) = \mathcal{E}_q \varphi_q(r) \tag{11}$$

where $L = l + (N-2)/2$ and where $l = 0, 1, 2, \ldots$ denotes the quantum number of the angular momentum, as usual. One has, in general, $N \geqslant 2$, but extrapolations to $N = 1$ can also be done. The pertinent quantum numbers are defined as

$$[[n]]_q = \frac{q^n - 1}{q - 1} \equiv q^{\frac{n-1}{2}} [n]_{\sqrt{q}}. \tag{12}$$

Choosing, for example, the harmonic oscillator and inserting $V(r) = \omega^2 r^2$ gives the q-deformed energy [12, 19]

$$\mathcal{E}_\gamma = \frac{\mu\omega}{(q+1)q^{d_0}}[[2d_0^O]]_q = \frac{\mu\omega}{q}[d_0^O]_q \tag{13}$$

where $d_0^O = l + 2n_r + N/2$ denotes the principal quantum number, whereas $n_r = 0, 1, 2, \ldots$ represents the radial quantum number. This result will be invoked in the next section. Other q-deformed radial equations can be readily established for related wavefunctions such as

$$\psi_q(r) = r^l f_q(r) = r^{l+\delta}\varphi_q(r) \tag{14}$$

where $q^\delta = (1 + q^{-2L})/(q+1)$.

3. Two-dimensional electrons under the influence of the magnetic field

The classical Hamiltonian describing two-dimensional electrons under the influence of a transversal and homogeneous magnetic field reads

$$\mathcal{H} = \frac{1}{2m_0}(\vec{p} + e\vec{A})^2 \tag{15}$$

where $e > 0$. We choose the vector potential in the symmetric gauge as

$$\vec{A} = \left(-\frac{B}{2}y, \frac{B}{2}x, 0\right). \tag{16}$$

Inserting the wavefunction

$$\Phi(\vec{x}) = \exp(\mathrm{i}m\varphi_0)\frac{\varphi(\rho)}{\sqrt{\rho}} \tag{17}$$

into the Schrödinger equation $\mathcal{H}\Phi = E\Phi$ gives the reduced dimensionless radial equation for an harmonic oscillator

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2} + \frac{m^2 - 1/4}{\xi^2}\varphi + \lambda_0^2\xi^2\varphi = \mathcal{E}\varphi \tag{18}$$

where $\rho = a\xi$, $\varphi(\xi) \in \{L_2(0, \infty), \mathrm{d}\xi\}$,

$$\mathcal{E} = \frac{2m_0 a^2}{\hbar^2}E^O \tag{19}$$

and

$$E^O = E - \frac{m}{2}\hbar\omega_c. \tag{20}$$

The cyclotron frequency is $\omega_c = eB/m_0$ and

$$\lambda_0^2 = \frac{e^2B^2a^4}{4\hbar^2} = \frac{a^4}{4l_C^2 l_O^2}. \tag{21}$$

The magnetic quantum number exhibits the values $m = 0, \pm1, \pm2, \ldots$, whereas ρ and φ_0 are the polar coordinates characterizing the classical (x, y)-plane. It is also clear that 'a' is an arbitrary length scale, $l_C = \hbar/m_0c$ denotes the Compton wavelength, while $l_O = c/\omega_c$ is a typical length characterizing the harmonic oscillator in equation (18). This latter equation is well known in the literature and general solutions have been written down in terms of Laguerre polynomials [20]. Now one has $L = |m|$ and $N = 2$, so that $l = |m|$. The principal quantum number is then given by

$$d_0 = d_0^O = |m| + 2n_r + 1 = 1, 2, 3, \ldots. \tag{22}$$

A standard harmonic oscillator, say $\omega_0^2\xi^2$, can also be inserted into equation (18), which amounts to the substitution

$$\omega_c^2 \to \Omega_c^2 = \omega_c^2 + 4\omega_0^2. \tag{23}$$

Using equations (13) and (22), we can then say that the q-deformed counterpart of E^O is

$$E_\gamma^O = \frac{\hbar}{2q}\Omega_c[d_0^O]_q = \frac{\hbar}{2}\Omega_c\frac{\sinh(\gamma d_0^O)}{q\sinh(\gamma)}. \tag{24}$$

It is understood that the q-deformation of the classical eigenfunction [20]

$$\varphi(\xi) = \left[\frac{2\Gamma(n_r+1)}{\Gamma(|m|+n_r+1)}\right]^{1/2} \lambda_0^{(|m|+1/2)/2}\exp\left(-\frac{\lambda_0}{2}\xi^2\right)L_{n_r}^{(|m|)}(\lambda_0\xi^2) \tag{25}$$

remains to be done in terms of q-Laguerre polynomials [15].

On the other hand, the Hamiltonian (15) has been discussed recently within the noncommutative θ-description [21]. This enables us to say that the θ-counterpart of E^O is given by

$$E_\theta^O = \frac{\hbar}{2}\Omega_c d_0^O\left[1 + \frac{\theta}{2l_Cl_O} + \left(\frac{\theta}{4l_Cl_O}\right)^2\Gamma_0\right]^{1/2} \tag{26}$$

by virtue of equation (32) in [21], where $\Gamma_0 = 1 + 4\omega_0^2/\omega_c^2$. This time the energy-splitting term has not been accounted for, as one deals selectively with the Hamiltonian of a radial harmonic oscillator. Alternatively, we can assume that $m = 0$, in which case the splitting term is ruled out from the very beginning. Now one realizes immediately that the matching condition

$$E_\theta^O = E_\gamma^O \tag{27}$$

is fulfilled if

$$\theta = \theta_\pm = \frac{4l_Cl_O}{\Gamma_0}[-1 \pm (1 - 2\gamma\Gamma_0)^{1/2}] \tag{28}$$

which works to first γ-order. Choosing $\theta = \theta_+$ gives

$$\theta = \theta_+ \cong -4\gamma l_Cl_O. \tag{29}$$

One would then obtain

$$x_2x_2 \cong 4l_Cl_O \tag{30}$$

by virtue of equation (7), which indicates that the quantum noncommutative plane should be located at

$$z = z_0 \cong 2\sqrt{l_Cl_O} \tag{31}$$

which represents an unexpected finding. An 'elementary length' could also be proposed via $\theta_+ \cong -l_0^2$, in which case

$$l_0 \cong \sqrt{\gamma}z_0 \tag{32}$$

which may be of interest from a general theoretical point of view.

4. Conclusions

In this paper we have discussed certain details concerning the description of electrons on a noncommutative plane threaded by a perpendicular and homogeneous magnetic field. For this purpose the noncommutative quantum Euclidean description has been analysed versus the noncommutative θ-description. Selecting the Hamiltonian with the quadratic interaction, we found that both descriptions produce the same energy if underlying deformation parameters, i.e. γ and θ, are inter-related, such as given by equation (28). We can then say that under the influence of noncommutativity, the principal quantum number d_0^O characterizing the equivalent two-dimensional harmonic oscillator becomes deformed as

$$d_0^O \to d_\gamma^O = \frac{1}{q}[d_0^O]_q \tag{33}$$

and

$$d_0^O \to d_\theta^O = d_0^O \sqrt{1 + \frac{\theta}{2l_C l_O} + \left(\frac{\theta}{4l_C l_O}\right)^2 \Gamma_0}. \tag{34}$$

In addition, we have also found that the location of the noncommutative plane becomes well established by virtue of equation (31), so that $z_0 \to \infty$ if $B \to 0$.

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